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Creation and annihilation operators for SU(3) in an SO(6, 2) model

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Abstract. Creation and annihilation operators are defined which are Wigner operators (tensor shift operators) for SU(3). While the annihilation operators are simply boson operators, the creation operators are cubic polynomials in boson operators. Together they generate under commutation the Lie algebra of SO(6, 2). The vector space generated from a vacuum vector by repeated application of the creation operators carries an irreducible representation of the SO(6, 2) algebra, equivalent to an hermitian representation, and also carries in direct sum every different irreducible representation of SU(3) < SO(6, 2) exactly once. A model for SU(3), in the sense of Bernšteĭn, Gel'fand and Gel'fand, is therefore defined. The different SU(3) irreducible representations appear explicitly as manifestly covariant, irreducible tensors, whose orthogonality and normalisation properties are examined. Other Wigner operators for SU(3) can be constructed simply as products of the new creation and annihilation operators, or sums of such products.

1. Introduction

The representation theory of the groups SU(n) continues to play an important role in several areas of quantum mechanics. While the theory has been most fully developed for SU(2) because of its association with angular momentum (see in particular Gel'fand *et al* (1963), Schwinger (1965) and Biedenharn and Louck (1981)), it is also true that many aspects have been extensively developed for larger values of n. In the present context, the works of Baird and Biedenharn (1963, 1964, 1965), Biedenharn *et al* (1967, 1972), Biedenharn and Louck (1968), Arisaka (1972), Holman and Biedenharn (1971), Louck and Biedenharn (1973) and Louck *et al* (1975) are particularly relevant.

The present work is concerned with the problem of constructing a simple *model* for SU(3), a group which occupies a favoured position in modern particle theory. Following Bernšteĭn *et al* (1975), a model of a compact Lie group G is defined as *a realisation of a representation of G which consists of a direct sum of irreducible representations (irreps), containing exactly one representative from every equivalence class of irreps of G.* A model of a group may be regarded as providing a minimal framework or skeleton for its representation theory.

Different models will exist for a given G. These will be equivalent as representations, but one model may have advantages over others for certain computational or explicative purposes because of its particular realisation.

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This is well illustrated in the case of SU(2), for which several models can be found in the literature quoted above, by reference to Schwinger's model (Schwinger 1965), which exploits the computational advantages of the boson calculus. Having introduced a pair of creation operators $\bar{\alpha}^r$ $(r=1,2)$, their Hermitian conjugate annihilation operators α_r, and a normalised 'vacuum vector' ϕ_0, one can easily construct basis vectors for each finite-dimensional irrep of SU(2), generated by (the hermitian linear combinations of) the operators

$$T^r_s=\bar{\alpha}^r\alpha_s-\tfrac{1}{2}\delta^r_s(\bar{\alpha}^t\alpha_t), \tag{1}$$

or equivalently, in a more familiar notation, by

$$J_1=\tfrac{1}{2}(\bar{\alpha}^1\alpha_2+\bar{\alpha}^2\alpha_1), \qquad J_2=\tfrac{1}{2}\mathrm{i}(\bar{\alpha}^2\alpha_1-\bar{\alpha}^1\alpha_2), \qquad J_3=\tfrac{1}{2}(\bar{\alpha}^1\alpha_1-\bar{\alpha}^2\alpha_2). \tag{2}$$

The two-boson Fock space $\mathscr{F}_2$ consisting of all finite linear combinations of vectors of the form $(\bar{\alpha}^1)^m(\bar{\alpha}^2)^n\phi_0$, with m and n non-negative integers, decomposes into a direct sum of SU(2)-irreducible subspaces. The basis vectors for the $(2j+1)$-dimensional subspace $\mathscr{V}_j$ on which the Casimir operator $\frac{1}{2}T^r_sT^s_r(=(J_1)^2+(J_2)^2+(J_3)^2)$ has the value $j(j+1)$, can be taken to be

$$\{(\bar{\alpha}^1)^{j+m}(\bar{\alpha}^2)^{j-m}/[(j+m)!(j-m)!]^{1/2}\}\phi_0, \qquad m=j,j-1,\ldots,-j, \tag{3}$$

corresponding to eigenvalues m of J_3. It can be seen that

$$\mathscr{F}_2=\mathscr{V}_0\oplus\mathscr{V}_{1/2}\oplus\mathscr{V}_1\oplus\ldots. \tag{4}$$

Thus every different irrep of SU(2) occurs exactly once in the representation generated by the T^r_s on $\mathscr{F}_2$, and a model is defined.

The operators α_r, $\bar{\alpha}^r$, A^r_s $(=\bar{\alpha}^r\alpha_s)$ and I (unit operator), or rather the nine independent Hermitian linear combinations of these operators, can be regarded as generators of a unitary irrep of a Lie group UW_2 with Lie algebra defined by the commutation relations

$$\begin{aligned}
&[\alpha_r,\alpha_s]=0=[\bar{\alpha}^r,\bar{\alpha}^s], \qquad [\alpha_r,\bar{\alpha}^s]=\delta^s_rI,\\
&[\alpha_r,I]=[\bar{\alpha}^r,I]=[A^r_s,I]=0,\\
&[\alpha_r,A^s_t]=\delta^s_r\alpha_t, \qquad [\bar{\alpha}^r,A^s_t]=-\delta^r_t\bar{\alpha}^s,\\
&[A^r_s,A^t_u]=\delta^t_sA^r_u-\delta^r_uA^t_s.
\end{aligned} \tag{5}$$

This group UW_2, which is neither compact nor semi-simple, has the Weyl–Heisenberg group W_2 as a subgroup, associated with the generators $\bar{\alpha}^r$, α_r and I; and U(2) as maximal compact subgroup, associated with the A^r_s. Schwinger's model of SU(2) may be regarded as defined by this unitary irrep of UW_2 on the closure of $\mathscr{F}_2$: when regarded as a representation of $\mathrm{SU}(2)<\mathrm{U}(2)<\mathrm{UW}_2$, it contains every different (unitary) irrep exactly once.

In attempting to generalise Schwinger's model to SU(3), one naturally considers at first the irrep of UW_3 on $\mathscr{F}_3$. Thus a triplet of boson pairs $\bar{\alpha}^r$, α_r $(r=1,2,3)$ can be introduced in place of the doublet used for SU(2), and SU(3) generators can be defined as

$$T^r_s=\bar{\alpha}^r\alpha_s-\tfrac{1}{3}\delta^r_s(\bar{\alpha}^t\alpha_t), \tag{6}$$

acting on the three-boson space $\mathscr{F}_3$. However, this does not define a model for SU(3) because, as is well known (Baird and Biedenharn 1963, 1964, 1965), not all irreps occur within the representation generated by these T^r_s (although those that do occur—the 'completely symmetric' ones, corresponding to one-rowed Young diagrams—occur once only). This can be remedied in (at least) two different ways.

One way, as expounded in detail by Baird and Biedenharn, is to introduce two (or more) triplets of boson pairs $\bar{\alpha}^r$, α_r, $\bar{\beta}^r$, β_r (here any α-operator commutes with any β-operator), and to define, in place of (6),

$$T^r_s = \bar{\alpha}^r\alpha_s - \tfrac{1}{3}\delta^r_s(\bar{\alpha}^t\alpha_t) + \bar{\beta}^r\beta_s - \tfrac{1}{3}\delta^r_s(\bar{\beta}^t\beta_t). \tag{7}$$

These generate on $\mathscr{F}_6$ a unitary representation of $SU(3) < UW_6$ which certainly contains every irrep at least once. Unfortunately a model is not thereby defined, because repetitions occur: for example, the vectors $\bar{\alpha}^r\phi_0$ and $\bar{\beta}^r\phi_0$ span distinct but equivalent three-dimensional irreps. This difficulty can be overcome, as Holman and Biedenharn (1971) have shown, by systematically restricting attention to a particular subspace of $\mathscr{F}_6$ which does contain each irrep of SU(3) exactly once, and so defines a model. Nevertheless, an attractive feature of Schwinger's model is now missing: the model does not admit every vector that can be obtained from the vacuum vector by application of the given creation operators. Furthermore, there are certainly many other ways of restricting $\mathscr{F}_6$ to a subspace that defines a model.

The second way of extending the three-boson representation of SU(3) to a model is again to introduce, together with $\bar{\alpha}^r$ and α_r, a second triplet of boson pairs, this time labelled $\bar{\beta}_r$, β^r, so that

$$[\alpha_r, \bar{\alpha}^s] = \delta^s_r = [\beta^s, \bar{\beta}_r], \qquad \alpha_r\phi_0 = \beta^r\phi_0 = 0, \tag{8}$$

where ϕ_0 is the vacuum vector. (All other commutators vanish.) The operators

$$T^r_s = \bar{\alpha}^r\alpha_s - \tfrac{1}{3}\delta^r_s(\bar{\alpha}^t\alpha_t) - \bar{\beta}_s\beta^r + \tfrac{1}{3}\delta^r_s(\bar{\beta}_t\beta^t) \tag{9}$$

replace those in (6) or (7), and satisfy the same commutation relations. Whereas the creation operators $\bar{\alpha}^r$ form a contravariant U(3)-vector, and the annihilation operators α_r a covariant U(3)-vector, the reverse is true for the creation operators $\bar{\beta}_r$ and annihilation operators β^r. Thus, having defined the U(3) generators

$$A^r_s = \bar{\alpha}^r\alpha_s - \bar{\beta}_s\beta^r, \tag{10}$$

which satisfy the usual relations

$$[A^r_s, A^t_u] = \delta^t_s A^r_u - \delta^r_u A^t_s, \tag{11}$$

one finds

$$[\bar{\alpha}^r, A^s_t] = -\delta^r_t\bar{\alpha}^s, \qquad [\alpha_r, A^s_t] = \delta^s_r\alpha_t, \tag{12}$$

but

$$[\bar{\beta}_r, A^s_t] = \delta^s_r\bar{\beta}_t, \qquad [\beta^r, A^s_t] = -\delta^r_t\beta^s. \tag{13}$$

This approach to SU(3) has been developed by Takabayasi (1964) and, more fully, by Arisaka (1972).

(The operators $\bar{\alpha}^r$, $\bar{\beta}_r$ may be thought of as 'quark' and 'anti-quark' creation operators, respectively, and correspond to conjugate three-dimensional irreps of U(3). Actually, the U(3) generators would be $A^r_s + \delta^r_s(\bar{\beta}_t\beta^t)$ rather than A^r_s as in (10), if the usual convention were adopted that the labelling of an irrep of U(3) by its highest weight should match exactly its labelling by the row lengths of a corresponding Young diagram. Then the vectors $\bar{\alpha}^r\phi_0$ and $\bar{\beta}_r\phi_0$ would correspond to irreps labelled (1, 0, 0) and $(1,1,0)$, whereas the generators (10) lead to a labelling by highest weights as $(1,0,0)$ and $(0,0,-1)$. The choice (10) and associated labelling by highest weights is more natural here and in what follows as it treats the α-operators and β-operators symmetrically, and leads to the simple relations (12) and (13).)

In the six-boson space $\mathscr{F}_6$ spanned by all the vectors that can be obtained from ϕ_0 by repeated application of the creation operators $\bar{\alpha}^r$ and $\bar{\beta}_r$, the SU(3) operators (9) generate a unitary representation that again contains every different irrep at least once. But again, a model is not defined because repetitions occur. For example, $(\bar{\alpha}^r\bar{\beta}_r)^N\phi_0$ is a singlet for every non-negative integer N. Arisaka has shown how to project onto SU(3)-irreducible subspaces, and it is therefore clear that various subspaces carrying every different irrep exactly once—and hence defining models—can be identified. Once again, however, there is no unique way to proceed, and whatever procedure is adopted, certain of the vectors obtainable by applying the boson creation operators to ϕ_0 have to be excluded from consideration.

Why does this complication arise for SU(3), in both the approaches described, but not for SU(2)? A partial answer is that, in the SU(2) case, the operators $\bar{\alpha}^r$, α_r $(r=1,2)$, are not only contravariant and covariant operators, respectively, with respect to U(2) and SU(2), they are also Wigner operators for both groups (Biedenharn and Louck 1981). That is to say, they are shift operators for the representation labels of these groups, and in particular for the SU(2) representation label j. When an $\bar{\alpha}^r$ is applied to a vector with a definite value of j, a vector is obtained with a value of j increased by one half unit. Thus one obtains from ϕ_0 $(j=0)$ the vectors $\bar{\alpha}^r\phi_0$ $(j=\frac{1}{2})$, $\bar{\alpha}^r\bar{\alpha}^s\phi_0$ $(j=1)$ etc, and automatically generates a chain of SU(2)-irreducible subspaces. The shifting property for the $\bar{\alpha}^r$ and α_r can be made explicit: the invariant $\frac{1}{2}T^r_sT^s_r$ for SU(2), with T^r_s as in (1), is found to satisfy

$$\tfrac{1}{2}T^r_sT^s_r=J(J+1), \qquad J=\tfrac{1}{2}\bar{\alpha}^r\alpha_r, \tag{14}$$

and J can be identified with the labelling operator whose eigenvalue is j. It is then evident from the boson commutation relations that

$$J\bar{\alpha}^r=\bar{\alpha}^r(J+\tfrac{1}{2}), \qquad J\alpha_r=\alpha_r(J-\tfrac{1}{2}). \tag{15}$$

In the approach of Baird and Biedenharn to U(3) and SU(3), the $\bar{\alpha}^r$ and $\bar{\beta}^r$ are three-vector operators, but they are not Wigner operators. For example, when $\bar{\beta}^r$ is applied to the vector $\bar{\alpha}^s\phi_0$, belonging to the U(3) irrep $(1,0,0)$, it produces a superposition of vectors in $(2,0,0)$ and $(1,1,0)$. Associated with this is the fact that one cannot find simple functions M and N of the boson operators $\bar{\alpha}^r$, α_r, $\bar{\beta}^r$ and β_r which have the eigenvalues m and n on the irrep labelled $(m,n,0)$. Similar remarks apply to the operators $\bar{\alpha}^r$, α_r, $\bar{\beta}_r$ and β^r of Arisaka's approach.

This suggests a possible way of avoiding the difficulty: replace the usual boson operators by creation and annihilation operators which *are* Wigner operators. In the spirit of Lohe and Hurst (1971), who defined 'modified' boson operators that are Wigner operators for the orthogonal and symplectic groups, and who used them, in effect, to formulate models for those groups, one may attempt to 'modify' in an appropriate way the boson operators of either the approach of Baird and Biedenharn or that of Arisaka. A general technique does exist for systematically resolving tensor operators of any classical group into Wigner operators (Bracken and Green 1971, Green 1971, Green and Bracken 1974, Gould 1980), and could be used here. However, one might expect that the Wigner operators obtained by this or some other method from the given boson operators would not be simply *polynomials* in those boson operators. (For instance, they are not polynomials in the cases treated by Lohe and Hurst.) Then their introduction might not provide an attractive resolution of the difficulties described above, because the computational simplicity associated with the boson calculus might be lost.

Fortunately it turns out that this is not the case for certain Wigner creation and annihilation operators for SU(3) which can be obtained very simply by modifying the boson operators of Arisaka's approach. These Wigner operators do have simple (cubic) expressions in terms of the boson operators, and they do satisfy simple algebraic relations. Furthermore, they have the fundamental property that all vectors (and only such vectors) obtainable from a vacuum vector by the application of these operators lie in a vector space which carries every different irrep of SU(3) exactly once. Their introduction therefore leads directly to a model for SU(3).

(After the preparation of the first version of this work, the authors' attention was drawn to preprints by Flath (1984), Flath and Biedenharn (1982), Biedenharn and Flath (1984a, b) in which some closely related results are obtained, but from a different direction. They arrive at the same realisation of the SO(6, 2) algebra described in §3 below, but their emphasis is on the algebra of SU(3) tensor operators and a resolution of the multiplicity problem for such operators, rather than the construction of creation and annihilation operators which lead to a model of SU(3). Sparling (1981) identified the relevant irrep of the SO(6, 2) algebra even earlier, and described some properties of its SU(3) content, within the context of a model of elementary particles. The authors are indebted to a referee for bringing this last reference to their attention.)

2. Wigner creation and annihilation operators

In the framework considered by Arisaka, the structure of the U(3) generators A^r_s as in (10) reflects a certain structure for that group's representation in the six-boson space $\mathscr{F}_6$: the $\bar{\alpha}^r\alpha_s$ are associated with a direct sum of irreps

$$\sum_{p=0}^{\infty} \oplus (p, 0, 0), \tag{16}$$

where p is the non-negative integral eigenvalue of the number operator

$$P = \bar{\alpha}^r\alpha_r; \tag{17}$$

and similarly the $-\bar{\beta}_s\beta^r$ are associated with a direct sum

$$\sum_{q=0}^{\infty} \oplus (0, 0, -q), \tag{18}$$

where q is the eigenvalue of

$$Q = \bar{\beta}_r\beta^r. \tag{19}$$

The A^r_s therefore generate a direct sum of representations, each of the form $(p, 0, 0)\otimes(0, 0, -q)$, which reduces as (Pais 1966)

$$(p, 0, 0)\otimes(0, 0, -q) = \sum_{k=0}^{m} \oplus (p-k, 0, k-q), \tag{20}$$

where m is the smaller of p and q. In full then, the A^r_s generate in $\mathscr{F}_6$ the representation

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=0}^{m}\oplus(p-k,0,k-q). \tag{21}$$

Consider the Casimir operator for U(3),

$$\tfrac{1}{2}A^r_sA^s_r=\tfrac{1}{2}(\bar{\alpha}^r\alpha_s-\bar{\beta}_s\beta^r)(\bar{\alpha}^s\alpha_r-\bar{\beta}_r\beta^s). \tag{22}$$

With the help of the boson commutation relations, it is easily checked that

$$\tfrac{1}{2}A^r_sA^s_r=\tfrac{1}{2}P(P+2)+\tfrac{1}{2}Q(Q+2)-X, \tag{23}$$

where X is the Hermitian operator

$$X=(\bar{\alpha}^r\bar{\beta}_r)(\alpha_s\beta^s). \tag{24}$$

It is known (Okubo 1962) that on an irrep of U(3) labelled (λ, μ, ν) by highest weights, the Casimir operator takes the value

$$\tfrac{1}{2}(\lambda^2+2\lambda+\mu^2+\nu^2-2\nu). \tag{25}$$

Therefore, on an irrep $(p-k,0,k-q)$ in the sum (21) one has

$$\tfrac{1}{2}A^r_sA^s_r=\tfrac{1}{2}p(p+2)+\tfrac{1}{2}q(q+2)-k(p+q+2-k), \tag{26}$$

and so

$$X=k(p+q+2-k). \tag{27}$$

It follows that, by restricting the representation space to the subspace $\mathscr{B}\subset\mathscr{F}_6$ on which $X=0$ (corresponding to $k=0$ in (27) and (21)), one restricts the representation of U(3) from that in (21) to

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(p,0,-q). \tag{28}$$

The operator X is the simplest of the 'trace' operators introduced by Arisaka and used by him to construct projectors onto U(3) irreps in the sum (21). Note that X has the form $\theta^\dagger\theta$, with $\theta=\alpha_r\beta^r$ and $\theta^\dagger$ its Hermitian conjugate. The condition that $X=0$ on $\mathscr{B}$ is therefore equivalent to the simpler condition

$$\alpha_r\beta^r\phi=0 \tag{29}$$

for every vector ϕ in $\mathscr{B}$.

Note that the typical representation $(p,0,-q)$ in the sum (28) is labelled by the eigenvalues p and q of P and Q, because it is associated unambiguously with the tensor product $(p,0,0)\otimes(0,0,-q)$. The number operators P and Q evidently form a complete set of independent U(3) scalars on $\mathscr{B}$ (though not on the larger space $\mathscr{F}_6$), and any other U(3) scalar is therefore, on $\mathscr{B}$, a function of them. In particular

$$\begin{aligned}&A^r_r=P-Q, \qquad A^r_sA^s_r=P(P+2)+Q(Q+2),\\&A^r_sA^s_tA^t_r=P^3-Q^3+4P^2+4P-2Q^2+2Q+PQ.\end{aligned} \tag{30}$$

These results may be verified explicitly, using the boson relations and the fact that (29) holds on $\mathscr{B}$, or they may be deduced from the known values (Okubo 1962) of these U(3) invariants on an irrep labelled $(p,0,-q)$.

Such an irrep remains irreducible when restricted to SU(3), and corresponds to the two-rowed Young diagram with row lengths $p+q$ and q: it will be labelled $(p+q, q)$ in what follows. Then it is a consequence of (28) that the subrepresentation of SU(3) generated in $\mathscr{B}$ by the operators T^r_s as in (9),

$$T^r_s = A^r_s - \tfrac{1}{3}\delta^r_s A^t_t, \tag{31}$$

has the form

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \oplus (p+q, q). \tag{32}$$

This sum can be seen to contain every different irrep of SU(3) exactly once.

Each term in this sum is associated in one-to-one fashion with a pair of eigenvalues p, q of P and Q. The independent SU(3) invariants on $\mathscr{B}$ take the form, from (30) and (31),

$$\begin{aligned} T^r_s T^s_r &= \tfrac{2}{3}(P^2+Q^2+PQ+3P+3Q), \\ T^r_s T^s_t T^t_r &= \tfrac{1}{9}(2P^3-2Q^3+3P^2Q-3PQ^2+18P^2+9PQ+36P+18Q). \end{aligned} \tag{33}$$

It is clear that any Wigner operator for SU(3) (or U(3)) on $\mathscr{B}$ must be a shift operator for P and Q, since it has to take a vector from one irrep to another, and hence one eigenvector of P and Q into another. Therefore it is of interest to construct, from the given boson operators, modified operators which shift the number operators P and Q while respecting the condition (29) which defines $\mathscr{B}$.

Consider the creation operator $\bar{\alpha}^r$, which raises the value of P by one unit and commutes with Q. Applied to a vector belonging to $(p, 0, -q)$ within $(p, 0, 0)\otimes(0, 0, -q)$, it must produce a vector contained in $(p+1, 0, 0)\otimes(0, 0, -q)$, that is to say, in the sum

$$\sum_{k=0}^{m'} (p+1-k, 0, k-q) \tag{34}$$

where m' is the smaller of $p+1$ and q. On the other hand, since $\bar{\alpha}^r$ transforms according to the irrep $(1, 0, 0)$, it must carry a vector belonging to $(p, 0, -q)$ into a vector belonging to $(p, 0, -q)\otimes(1, 0, 0)$, which reduces as (Pais 1966)

$$(p, 0, -q)\otimes(1, 0, 0) = (p+1, 0, -q)\oplus(p, 1, -q)\oplus(p, 0, 1-q). \tag{35}$$

A comparison of (34) and (35) shows that in general $\bar{\alpha}^r$ carries a vector from $(p, 0, -q)$ within $\mathscr{B}$ into $(p+1, 0, -q)\oplus(p, 0, 1-q)$. But the resultant vector is an eigenvector of P and Q with eigenvalues $p+1$ and q, and the only vectors in $\mathscr{B}$ corresponding to such eigenvalues lie in $(p+1, 0, -q)$. Therefore $\bar{\alpha}^r$ is a sum of two operators. One, say $\bar{\alpha}^{(1)r}$, shifts $(p, 0, -q)$ in $\mathscr{B}$ into $(p+1, 0, -q)$ in $\mathscr{B}$; the other, say $\bar{\alpha}^{(2)r}$, shifts $(p, 0, -q)$ in $\mathscr{B}$ into $(p, 0, 1-q)$ outside $\mathscr{B}$. Now formulae (23) and (25) show not only that $X=0$ on $\mathscr{B}$ but also that $X=p+q+2$ on $(p+1, 0, 1-q)$ within $(p+1, 0, 0)\otimes(0, 0, -q)$. It then follows that, on vectors in $\mathscr{B}$,

$$X\bar{\alpha}^{(1)r} = \bar{\alpha}^{(1)r}X, \qquad (X-P-Q-1)\bar{\alpha}^{(2)r} = \bar{\alpha}^{(2)r}X. \tag{36}$$

Since

$$\bar{\alpha}^r = \bar{\alpha}^{(1)r} + \bar{\alpha}^{(2)r} \tag{37}$$

it follows from (36) that, on $\mathscr{B}$,

$$(P+Q+1)\bar{\alpha}^r+[\bar{\alpha}^r, X]=(P+Q+1)\bar{\alpha}^{(1)r}. \tag{38}$$

This operator $(P+Q+1)\bar{\alpha}^{(1)r}$ has the desired shifting properties, prompting the definition of the modified creation operator

$$\bar{A}^r=(P+Q+1)\bar{\alpha}^r+[\bar{\alpha}^r, X]=(P+Q+1)\bar{\alpha}^r-(\bar{\alpha}^s\bar{\beta}_s)\beta^r. \tag{39}$$

Then $\bar{A}^r$ is a Wigner operator on $\mathscr{B}$, carrying $(p, 0, -q)$ into $(p+1, 0, -q)$ while raising the eigenvalue of P by one unit and commuting with Q. Similarly, define

$$\bar{B}_r=(P+Q+1)\bar{\beta}_r-(\bar{\alpha}^s\bar{\beta}_s)\alpha_r. \tag{40}$$

It also is a Wigner operator on $\mathscr{B}$, carrying $(p, 0, -q)$ into $(p, 0, -q-1)$ while raising the eigenvalue of Q by one unit and commuting with P. It can now be checked directly that $\bar{A}^r$ and $\bar{B}_r$ leave the condition (29), and hence the subspace $\mathscr{B}$, invariant: for example the commutator of $\bar{A}^r$ with $\alpha_s\beta^s$ equals $-2\bar{\alpha}^r(\alpha_s\beta^s)$ and hence vanishes on $\mathscr{B}$.

The annihilation operators α_r and β_r leave invariant the condition (29) and hence the subspace $\mathscr{B}$, and they require no modification. Since they are shift operators for P and Q they are also Wigner operators on $\mathscr{B}$. In fact, within $\mathscr{B}$, α_r carries $(p, 0, -q)$ into $(p-1, 0, -q)$ while β^r carries it into $(p, 0, 1-q)$.

Accordingly, take as conjugate to $\bar{A}^r$ and $\bar{B}_r$ above,

$$A_r=\alpha_r, \qquad B^r=\beta^r. \tag{41}$$

Then A_r is not *Hermitian* conjugate to $\bar{A}^r$. In fact

$$\bar{A}^{r\dagger}=\alpha_r(P+Q+1)-\bar{\beta}_r(\alpha_s\beta^s) \tag{42}$$

which reduces to $\alpha_r(P+Q+1)(=(P+Q+2)\alpha_r)$ on $\mathscr{B}$ because of (29). On $\mathscr{B}$ then,

$$\bar{A}^{r\dagger}=(P+Q+2)A_r \tag{43}$$

and similarly

$$\bar{B}_r^\dagger=(P+Q+2)B^r. \tag{44}$$

However these operators are related by a similarity transformation on $\mathscr{B}$ to operators $\bar{A}^{r\prime}$, $\bar{B}'_r$ and their Hermitian conjugates A'_r, $B^{r\prime}$. To see this, consider the Hermitian operator S which, when applied to any eigenvector of P and Q with eigenvalues p and q, takes the value $[(p+q+1)!]^{1/2}$. Symbolically

$$S(P, Q)=[(P+Q+1)!]^{1/2}. \tag{45}$$

This operator has a well defined inverse, which may be written as

$$S(P, Q)^{-1}=[(P+Q+1)!]^{-1/2}. \tag{46}$$

Define

$$\bar{A}^{r\prime}=S(P, Q)^{-1}\bar{A}^rS(P, Q)=\bar{A}^rS(P+1, Q)^{-1}S(P, Q)=\bar{A}^r(P+Q+2)^{-1/2} \tag{47}$$

and similarly

$$\begin{aligned}\bar{B}'_r&=S(P, Q)^{-1}\bar{B}_rS(P, Q)=\bar{B}_r(P+Q+2)^{-1/2},\\ A'_r&=S(P, Q)^{-1}A_rS(P, Q)=(P+Q+2)^{1/2}A_r,\\ B^{r\prime}&=S(P, Q)^{-1}B^rS(P, Q)=(P+Q+2)^{1/2}B^r.\end{aligned} \tag{48}$$

It then follows from (43) that, on $\mathscr{B}$,

$$\bar{A}^{r\prime\dagger} = (P+Q+2)^{-1/2}\bar{A}^{r\dagger} = (P+Q+2)^{1/2}A_r = A'_r \tag{49}$$

and similarly

$$\bar{B}'^{\dagger}_r = B^{r\prime}. \tag{50}$$

The primed operators satisfy the same commutation relations as the unprimed ones, and they are also Wigner operators for U(3) on $\mathscr{B}$, with similar shifting properties for P and Q. However, the unprimed operators are preferred in what follows because of their simpler expressions and consequent easier manipulation: the unprimed, and not the primed operators, are simply polynomials in the boson operators. Their unusual conjugacy properties cause no difficulties, as will be seen.

3. A realisation of SO(6, 2)

The modified creation and annihilation operators introduced in (39), (40) and (41) generate under commutation a representation of the Lie algebra of the simple Lie group SO(6, 2). To see this, introduce the number operators P and Q, and the U(3) generators A^r_s as in (17), (19) and (10), and let

$$\begin{aligned} &T_{rs} = \alpha_r\bar{\beta}_s - \alpha_s\bar{\beta}_r = -T_{sr}, \qquad T^{rs} = \beta^r\bar{\alpha}^s - \beta^s\bar{\alpha}^r = -T^{sr},\\ &M = P+Q+2. \end{aligned} \tag{51}$$

Note that T^{rs} is Hermitian conjugate to T_{sr}. It is straightforward to verify the following commutation relations:

$$\begin{aligned}
&[\bar{A}^r, \bar{A}^s] = [\bar{A}^r, \bar{B}_s] = [\bar{B}_r, \bar{B}_s] = 0, \qquad [A_r, A_s] = [A_r, B^s] = [B^r, B^s] = 0,\\
&[A_r, \bar{A}^s] = \delta^s_r M + A^s_r, \qquad [B^s, \bar{B}_r] = \delta^s_r M - A^s_r,\\
&[\bar{A}^r, B^s] = T^{rs}, \qquad [A_r, \bar{B}_s] = T_{rs},\\
&[\bar{A}^r, A^s_t] = -\delta^r_t\bar{A}^s, \qquad [\bar{B}_r, A^s_t] = \delta^s_r\bar{B}_t,\\
&[A_r, A^s_t] = \delta^s_r A_t, \qquad [B^r, A^s_t] = -\delta^r_t B^s,\\
&[\bar{A}^r, T_{st}] = \delta^r_t\bar{B}_s - \delta^r_s\bar{B}_t, \qquad [A_r, T_{st}] = 0,\\
&[\bar{B}_r, T_{st}] = 0, \qquad [B^r, T_{st}] = \delta^r_t A_s - \delta^r_s A_t,\\
&[\bar{A}^r, T^{st}] = 0, \qquad [A_r, T^{st}] = \delta^t_r B^s - \delta^s_r B^t,\\
&[\bar{B}_r, T^{st}] = \delta^t_r\bar{A}^s - \delta^s_r\bar{A}^t, \qquad [B^r, T^{st}] = 0,\\
&[\bar{A}^r, M] = -\bar{A}^r, \qquad [A_r, M] = A_r,\\
&[\bar{B}_r, M] = -\bar{B}_r, \qquad [B^r, M] = B^r,\\
&[A^r_s, A^t_u] = \delta^t_s A^r_u - \delta^r_u A^t_s, \qquad [A^r_s, T_{tu}] = -\delta^r_t T_{su} - \delta^r_u T_{ts},\\
&[A^r_s, T^{tu}] = \delta^t_s T^{ru} + \delta^u_s T^{tr}, \qquad [A^r_s, M] = 0,\\
&[T_{rs}, T_{tu}] = 0 = [T^{rs}, T^{tu}], \qquad [T_{rs}, T^{tu}] = \delta^t_r A^u_s + \delta^u_s A^t_r - \delta^t_s A^u_r - \delta^u_r A^t_s,\\
&[T_{rs}, M] = 0 = [T^{rs}, M].
\end{aligned} \tag{52}$$

Of the operators $\bar{A}^r$, A_r, $\bar{B}_r$, B^r, A^r_s, T_{rs}, T^{rs} and M, 28 are linearly independent. Now define an equivalent set of operators J_{AB} $(=-J_{BA})$ for A, $B=1,2,\ldots,8$, by setting

$$\begin{aligned}
&J_{2r-1,2s-1}=-\tfrac{1}{2}\mathrm{i}(T_{rs}+A^r_s-A^s_r+T^{rs}), \qquad J_{2r-1,2s}=-\tfrac{1}{2}(T_{rs}+A^r_s+A^s_r-T^{rs}),\\
&J_{2r,2s}=\tfrac{1}{2}\mathrm{i}(T_{rs}-A^r_s+A^s_r+T^{rs}),\\
&J_{7,2r-1}=-\tfrac{1}{2}(A_r+\bar{A}^r-\bar{B}_r-B^r), \qquad J_{7,2r}=\tfrac{1}{2}\mathrm{i}(A_r-\bar{A}^r-\bar{B}_r+B^r),\\
&J_{8,2r-1}=\tfrac{1}{2}\mathrm{i}(A_r-\bar{A}^r+\bar{B}_r-B^r), \qquad J_{8,2r}=\tfrac{1}{2}(A_r+\bar{A}^r+\bar{B}_r+B^r), \qquad J_{78}=M,
\end{aligned}\tag{53}$$

for r, $s=1,2,3$. Then the commutation relations (52) assume the familiar form for SO(6, 2),

$$[J_{AB}, J_{CD}]=\mathrm{i}(g_{AC}J_{BD}+g_{BD}J_{AC}-g_{BC}J_{AD}-g_{AD}J_{BC}) \tag{54}$$

where the metric tensor $g_{AB}=\operatorname{diag}(1,1,1,1,1,1,-1,-1)$.

Note the compact SO(6) subgroup associated with the Hermitian operators J_{ab}, $a,b=1,2,\ldots,6$, or equivalently, with the Hermitian linear combinations of the T_{rs}, T^{rs} and A^r_s; and the maximal compact SO(6)⊗SO(2) subgroup associated with this set of operators enlarged by the addition of $J_{78}(=M)$. Note also the U(3) < SO(6) subgroup associated as before with the operators A^r_s of (10), and the subgroup SU(3) < U(3) < SO(6) < SO(6, 2), associated as before with the T^r_s of (9).

The relations inverse to (53) are given by

$$\begin{aligned}
&\bar{A}^r=-\tfrac{1}{2}(J_{7,2r-1}-\mathrm{i}J_{7,2r}-\mathrm{i}J_{8,2r-1}-J_{8,2r}),\\
&A_r=-\tfrac{1}{2}(J_{7,2r-1}+\mathrm{i}J_{7,2r}+\mathrm{i}J_{8,2r-1}-J_{8,2r}),\\
&\bar{B}_r=\tfrac{1}{2}(J_{7,2r-1}+\mathrm{i}J_{7,2r}-\mathrm{i}J_{8,2r-1}+J_{8,2r}), \qquad B^r=\tfrac{1}{2}(J_{7,2r-1}-\mathrm{i}J_{7,2r}+\mathrm{i}J_{8,2r-1}+J_{8,2r}),\\
&A^r_s=\tfrac{1}{2}(\mathrm{i}J_{2r-1,2s-1}-J_{2r-1,2s}+J_{2r,2s-1}+\mathrm{i}J_{2r,2s}),\\
&T_{rs}=\tfrac{1}{2}(\mathrm{i}J_{2r-1,2s-1}-J_{2r-1,2s}-J_{2r,2s-1}-\mathrm{i}J_{2r,2s}),\\
&T^{rs}=\tfrac{1}{2}(\mathrm{i}J_{2r-1,2s-1}+J_{2r-1,2s}+J_{2r,2s-1}-\mathrm{i}J_{2r,2s}), \qquad M=J_{78},
\end{aligned}\tag{55}$$

for r, $s=1,2,3$.

The operators J_{AB} define a representation of the Lie algebra of SO(6, 2) in the whole of the six-boson Fock space $\mathscr{F}_6$ generated from ϕ_0 by the action of the $\bar{\alpha}^r$ and $\bar{\beta}_r$. Consider instead the subspace $\mathscr{B}'\subset\mathscr{F}_6$ which is generated from ϕ_0 by the application of arbitrary finite polynomials in the modified operators, $\bar{A}^r$, A_r, $\bar{B}_r$, B^r. This subspace is invariant under the action of the SO(6, 2) operators J_{AB}, and so carries itself a subrepresentation of the SO(6, 2) Lie algebra. In order to see this, it suffices to note that the $\bar{A}^r$, A_r, $\bar{B}_r$ and B^r leave $\mathscr{B}'$ invariant as a result of its definition, and also that they generate the whole of the SO(6, 2) algebra under commutation, as the relations (52) show.

The condition (29) holds for every vector ϕ in $\mathscr{B}'$. This follows because ϕ_0 satisfies the condition, and the modified creation and annihilation operators which generate $\mathscr{B}'$ from ϕ_0 leave the condition invariant, as was shown in § 2. It follows that $\mathscr{B}'$ is a subspace of the space $\mathscr{B}$ introduced there, but in fact the two spaces carry the same representation of U(3) and so are one and the same, as will be shown in § 4.

Since the A_r, B^r are not Hermitian conjugate to $\bar{A}^r$, $\bar{B}_r$ the representation of the SO(6, 2) algebra defined by (53) is not Hermitian: the operators J_{7a} and J_{8a}, $a=1,2,\ldots,6$ are not Hermitian. (Note that the representation of the SO(6)⊗SO(2) subalgebra is Hermitian because T_{rs} and T^{sr} are Hermitian conjugate, as are A^r_s and

A^s_r.) However, the restriction of this representation to $\mathscr{B}'$ is equivalent to an Hermitian one because the algebra is generated by $(A_r, \bar{A}^r)$ and $(B^r, \bar{B}_r)$, and these are similar on $\mathscr{B}'$ to Hermitian conjugate pairs $(A'_r, \bar{A}'^r)$, $(B'^r, \bar{B}'_r)$, as shown previously. (It does not follow that the representation on the larger space $\mathscr{F}_6$ is equivalent to an Hermitian representation.)

How can the representation of the SO(6, 2) algebra on $\mathscr{B}'$ be characterised? It is not hard to see, as will be noted in § 4, that it is irreducible. Furthermore it is reasonably straightforward to check that the Casimir operator

$$\tfrac{1}{2}g^{AC}g^{BD}J_{AB}J_{CD}, \tag{56}$$

where $g^{AB} = g_{AB}$, has the value -8 on $\mathscr{B}'$. This is not, of course, enough to identify the representation, but the other three independent invariants (one is a polynomial of degree six in the J_{AB}) have not been evaluated.

If the free indices in (53) are restricted to run over 1, 2 rather than 1, 2, 3, the subscripts 7 and 8 in (53) are replaced by 5 and 6, the factor $(P+Q+1)$ in (39) and (40) is replaced by $(P+Q)$, and $(P+Q+2)$ in (51) is replaced by $(P+Q+1)$, then one obtains a representation of the Lie algebra of SO(4, 2), which may then be restricted to the subspace of $\mathscr{F}_4$ on which $\alpha_r\beta^r = 0$. This sub-representation is analogous to the representation of the SO(6, 2) algebra on $\mathscr{B}'$, and is identifiable as equivalent to one of the well known 'ladder' representations of the SO(4, 2) algebra. (It corresponds to $\lambda = 0$ in the notation of Mack and Todorov (1969), and belongs to the $\mathscr{L}$ or $\mathscr{L}^*$ series, depending on whether the J_{AB} are defined just as in (53), or have J_{56} and J_{6a}, $a = 1, 2, 3, 4$, replaced therein by their negatives.) This follows because the operators J_{AB} ($A, B = 1, 2, \ldots, 6$) in that SO(4, 2) subrepresentation can be shown to satisfy the relations

$$J_{AC}J^C{}_B + J_{BC}J^C{}_A = -\tfrac{1}{3}g_{AB}J_{CD}J^{CD}, \qquad \tfrac{1}{2}J_{AB}J^{AB} = -3 \tag{57}$$

where $J^C{}_B = g^{CD}J_{DB}$ etc, and $g^{AB} = g_{AB} = \mathrm{diag}(1, 1, 1, 1, -1, -1)$. Such relations are known to characterise these ladder representations of SO(4, 2) (Barut and Böhm 1970).

Note that what is involved here is not the standard realisation of a ladder representation on $\mathscr{F}_4$. Some of the J_{AB} are cubic in the boson operators, whereas they are all quadratic in the standard realisation.

These results in the SO(4, 2) case suggest that in the SO(6, 2) case, it may be possible to find another realisation, with each J_{AB} a quadratic expression in boson operators. (However, the number of boson pairs needed may exceed six.) The modified creation and annihilation operators would then have a quadratic realisation in terms of boson operators. This interesting possibility is being explored (Bracken 1984).

In concluding this section, note that the definitions (39), (40) and (41) imply

$$\bar{A}^r\bar{B}_r = \bar{B}_r\bar{A}^r = (\bar{\alpha}^s\bar{\beta}_s)^2(\alpha_r\beta^r), \qquad A_rB^r = B^rA_r = \alpha_r\beta^r. \tag{58}$$

Therefore the following relations hold on $\mathscr{B}'$:

$$\bar{A}^r\bar{B}_r = \bar{B}_r\bar{A}^r = A_rB^r = B^rA_r = 0. \tag{59}$$

The first two of these will play a key role in § 4.

It also follows from the definitions given that

$$\begin{aligned} \bar{A}^rA_r - P(P+Q+1) &= A_r\bar{A}^r - (P+2)(P+Q+3) = -(\bar{\alpha}^s\bar{\beta}_s)(\alpha_r\beta^r), \\ \bar{B}_rB^r - Q(P+Q+1) &= B^r\bar{B}_r - (Q+2)(P+Q+3) = -(\bar{\alpha}^s\bar{\beta}_s)(\alpha_r\beta^r), \end{aligned} \tag{60}$$

so that

$$P = \tfrac{1}{3}[A_r, \bar{A}^r] - \tfrac{1}{6}[B^r, \bar{B}_r] - 1, \qquad Q = \tfrac{1}{3}[B^r, \bar{B}_r] - \tfrac{1}{6}[A_r, \bar{A}^r] - 1, \tag{61}$$

and hence $(P+1)$ and $(Q+1)$, but not the number operators P and Q themselves, are contained in the SO(6, 2) Lie algebra. Equations (60) also show that, on $\mathscr{B}'$,

$$\begin{aligned} \bar{A}^r A_r &= P(P+Q+1), & \bar{B}_r B^r &= Q(P+Q+1), \\ A_r\bar{A}^r &= (P+2)(P+Q+3), & B^r\bar{B}_r &= (Q+2)(P+Q+3). \end{aligned} \tag{62}$$

4. Irreducible tensor representations of SU(3)

For fixed integers $p \geqslant 0$ and $q \geqslant 0$, consider all vectors of the form

$$\phi^{rs\ldots t}_{kl\ldots m} = \bar{A}^r\bar{A}^s\ldots\bar{A}^t\bar{B}_k\bar{B}_l\ldots\bar{B}_m\phi_0 \tag{63}$$

in which p of the $\bar{A}$-operators and q of the $\bar{B}$ operators appear, and the indices run over 1 to 3 independently. Each such vector evidently lies in $\mathscr{B}'$ and is an eigenvector of P and Q with eigenvalues p and q. Note from the relations (52) that the order of the $\bar{A}$- and $\bar{B}$-operators here is immaterial. From the commutation relations (52) and the fact that $A^r_s\phi_0 = 0$, it follows that

$$\begin{aligned} A^u_n\phi^{rs\ldots t}_{kl\ldots m} &= \delta^r_n\phi^{us\ldots t}_{kl\ldots m} + \delta^s_n\phi^{ru\ldots t}_{kl\ldots m} + \ldots + \delta^t_n\phi^{rs\ldots u}_{kl\ldots m} \\ &\quad - \delta^u_k\phi^{rs\ldots t}_{nl\ldots m} - \delta^u_l\phi^{rs\ldots t}_{kn\ldots m} - \ldots - \delta^u_m\phi^{rs\ldots t}_{kl\ldots n}, \end{aligned} \tag{64}$$

so that $\phi^{rs\ldots t}_{kl\ldots m}$ transforms in a manifestly covariant way, as a U(3) tensor. This tensor is irreducible, corresponding to the representation $(p, 0, -q)$ of U(3). In order to see this, note from the relations (52) that ϕ is separately symmetric in its upper and lower indices, and from the relations (59) that

$$\phi^{rs\ldots t}_{rl\ldots m} = (\bar{A}^r\bar{B}_r)\bar{A}^s\ldots\bar{A}^t\bar{B}_l\ldots\bar{B}_m\phi_0 = 0. \tag{65}$$

It is known that a tensor satisfying these two conditions is U(3) and SU(3) irreducible, corresponding to the irrep $(p+q, q)$ of SU(3), and hence to an irrep of U(3) labelled $(p+r, r, r-q)$ for some r (Pais 1966). Since it follows from (64) that

$$A^n_n\phi^{rs\ldots t}_{kl\ldots m} = (p-q)\phi^{rs\ldots t}_{kl\ldots m}, \tag{66}$$

and since it is known (Okubo 1962) that, on $(p+r, r, r-q)$,

$$A^n_n = (p-q+3r), \tag{67}$$

it follows that $r=0$ and that $\phi^{rs\ldots t}_{kl\ldots m}$ corresponds to the U(3) irrep $(p, 0, -q)$. It can be seen now that every different irrep of SU(3) occurs just once as p and q in (63) run over the non-negative integers independently.

Every vector in $\mathscr{B}'$ can be written as a finite linear combination of vectors of the form (63). This follows from the definition of $\mathscr{B}'$, the commutation relations (52), and the fact that, as is easily checked from (8) and the definitions above,

$$(M-2)\phi_0 = A_r\phi_0 = B^r\phi_0 = 0, \qquad T_{rs}\phi_0 = T^{rs}\phi_0 = A^r_s\phi_0 = 0. \tag{68}$$

For example, consider the vector $A_r\bar{B}_s\bar{A}^t\phi_0$:

$$\begin{aligned} A_r\bar{B}_s\bar{A}^t\phi_0 &= \bar{B}_sA_r\bar{A}^t\phi_0 + T_{rs}\bar{A}^t\phi_0 \\ &= \bar{B}_s\bar{A}^tA_r\phi_0 + \bar{B}_s(\delta^t_rM + A^t_r)\phi_0 + \bar{A}^tT_{rs}\phi_0 + (\delta^t_r\bar{B}_s - \delta^t_s\bar{B}_r)\phi_0 \\ &= 3\delta^t_r\bar{B}_r\phi_0 - \delta^t_s\bar{B}_r\phi_0 = 3\delta^t_r\phi_s - \delta^t_s\phi_r \end{aligned} \tag{69}$$

It then follows that $\mathscr{B}'$ has the same U(3) content (28) as $\mathscr{B}$, and since it has already been shown that $\mathscr{B}'$ is a subspace of $\mathscr{B}$, it can be concluded that these two subspaces are indeed one and the same. The notation $\mathscr{B}'$ will henceforth be dropped.

Since $\bar{A}^r$ carries the U(3) irrep $(p, 0, -q)$ into $(p+1, 0, -q)$, it is clear that there is no proper subspace of $\mathscr{B}$ which is both U(3) invariant, and invariant under the action of $\bar{A}^r$. It follows, *a fortiori*, that there is no proper subspace invariant under the action of the whole SO(6, 2) algebra (53), so the representation of this algebra on $\mathscr{B}$ is indeed irreducible, as suggested earlier.

What are the orthogonality properties and lengths of the vectors (63)? The scalar product

$$(\phi^{rs\ldots t}_{kl\ldots m}, \phi^{\rho\sigma\ldots\tau}_{\kappa\lambda\ldots\mu}) \tag{70}$$

can of course be calculated in any particular case by writing the $\bar{A}$- and $\bar{B}$- operators, in expressions like (63), in terms of boson operators, using the definitions (39) and (40), and by then applying the usual boson calculus. However, the problem can also be approached somewhat more directly as follows.

Suppose the left-hand member of the scalar product (70) has p upper and q lower indices, while the right-hand member has p' upper and q' lower indices. Then the scalar product vanishes unless $p = p'$ and $q = q'$, because the two members are eigenvectors of the Hermitian operators P and Q with eigenvalues (p, q), (p', q') respectively. When each member has p upper and q lower indices, and so corresponds to the irrep $(p, 0, -q)$ of U(3), it follows by covariance that the scalar product must be a multiple of the numerical tensor (a linear combination of products of Kronecker deltas)

$$D^{(\rho\sigma\ldots\tau);[kl\ldots m]}_{[rs\ldots t];(\kappa\lambda\ldots\mu)} = \delta^\rho_r\delta^\sigma_s\ldots\delta^\tau_t\delta^k_\kappa\delta^l_\lambda\ldots\delta^m_\mu+\ldots, \tag{71}$$

whose definition is completed by the requirement that it is separately symmetric in each of the bracketed sets of indices, and vanishes if any round-bracketed (resp. square-bracketed) upper index is contracted with any round-bracketed (resp. square-bracketed) lower index. For example

$$D^{(\rho\sigma);[k]}_{[rs];(\kappa)} = \delta^\rho_r\delta^\sigma_s\delta^k_\kappa+\delta^\rho_s\delta^\sigma_r\delta^k_\kappa-\tfrac{1}{4}(\delta^\rho_s\delta^\sigma_\kappa\delta^k_r+\delta^\rho_\kappa\delta^\sigma_s\delta^k_r+\delta^\rho_r\delta^\sigma_\kappa\delta^k_s+\delta^\rho_\kappa\delta^\sigma_r\delta^k_s). \tag{72}$$

Then, if both tensors belong to $(p, 0, -q)$,

$$(\phi^{rs\ldots t}_{kl\ldots m}, \phi^{\rho\sigma\ldots\tau}_{\kappa\lambda\ldots\mu}) = \theta(p, q)D^{(\rho\sigma\ldots\tau);[kl\ldots m]}_{[rs\ldots t];(\kappa\lambda\ldots\mu)} \tag{73}$$

with $\theta(p, q)$ a number depending only on p and q.

Consider

$$\chi = \phi^{33\ldots3}_{33\ldots3} = (\bar{A}^3)^p(\bar{B}_3)^q\phi_0. \tag{74}$$

It follows from (43) and (44) and the shifting properties of A_r and B^r that the Hermitian conjugate of $(\bar{A}^3)^p(\bar{B}_3)^q$ is

$$\begin{aligned}(P+Q+2)B^3(P+Q+2)B^3\ldots(P+Q+2)B^3(P+Q+2)A_3\ldots(P+Q+2)A_3\\ =(P+Q+2)(P+Q+3)\ldots(P+Q+p+q+1)(B^3)^q(A_3)^p,\end{aligned} \tag{75}$$

so that

$$\begin{aligned}\|\chi\|^2 = (\chi, \chi) &= (\phi_0, (P+Q+2)\ldots(P+Q+p+q+1)(B^3)^q(A_3)^p(\bar{A}^3)^p(\bar{B}_3)^q\phi_0)\\ &=(p+q+1)!(\phi_0, (B^3)^q(A_3)^p(\bar{A}^3)^p(\bar{B}_3)^q\phi_0).\end{aligned} \tag{76}$$

For any vector of the form χ, one has

$$A_3^3\chi = (A_r^r - A_1^1 - A_2^2)\chi = A_r^r\chi = (P-Q)\chi, \tag{77}$$

using (52). Then, also with the help of these relations (52), one has

$$\begin{aligned}[A_3\bar{A}^3 - (P+1)(P+2), \bar{A}^3]\chi &= \{[A_3, \bar{A}^3]\bar{A}^3 - 2(P+1)\bar{A}^3\}\chi \\ &= (Q - P + A_3^3)\bar{A}^3\chi = 0.\end{aligned} \tag{78}$$

Furthermore, again from (52),

$$[A_3\bar{A}^3 - (P+1)(P+2), \bar{B}_3]\chi = 0. \tag{79}$$

Since $\{A_3\bar{A}^3 - (P+1)(P+2)\}$ vanishes on ϕ_0, it then follows that it vanishes on any vector of the form χ. Similarly $\{B^3\bar{B}_3 - (Q+1)(Q+2)\}$ vanishes on such vectors. Thus $A_3\bar{A}^3$ and $B^3\bar{B}_3$ can be replaced by $(P+1)(P+2)$ and $(Q+1)(Q+2)$, respectively, on such vectors. Then, from (76),

$$\begin{aligned}\|\chi\|^2 &= (p+q+1)!(\phi_0, (B^3)^q(A_3)^{p-1}(A_3\bar{A}^3)(\bar{A}^3)^{p-1}(\bar{B}_3)^q\phi_0) \\ &= (p+q+1)!(p+1)p(\phi_0, (B^3)^q(A_3)^{p-1}(\bar{A}^3)^{p-1}(\bar{B}_3)^q\phi_0) \\ &= (p+q+1)!(p+1)!p!(\phi_0, (B^3)^{q-1}(B^3\bar{B}_3)(\bar{B}_3)^{q-1}\phi_0) \\ &= (p+q+1)!(p+1)!p!(q+1)!q!.\end{aligned} \tag{80}$$

Now (73) implies that

$$\theta(p, q) = \|\phi^{33\ldots3}_{33\ldots3}\|^2 / D^{(33\ldots3);[33\ldots3]}_{[33\ldots3];(33\ldots3)} \tag{81}$$

so that one has, finally,

$$(\phi^{rs\ldots t}_{kl\ldots m}, \phi^{\rho\sigma\ldots\tau}_{\kappa\lambda\ldots\mu}) = (p+q+1)!(p+1)!p!(q+1)!q!\,D^{(\rho\sigma\ldots\tau);[kl\ldots m]}_{[rs\ldots t];(\kappa\lambda\ldots\mu)} / D^{(33\ldots3);[33\ldots3]}_{[33\ldots3];(33\ldots3)}. \tag{82}$$

For example, because (72) implies

$$D^{(33);[3]}_{[33];(3)} = 1, \tag{83}$$

it follows that

$$(\phi^{rs}_{k}, \phi^{\rho\sigma}_{\kappa}) = 4!3!2!2!1!\,D^{(\rho\sigma);[k]}_{[rs];(\kappa)}. \tag{84}$$

5. Concluding remarks

A model of SU(3) has been constructed in the space $\mathscr{B}$ obtained by applying arbitrary polynomials in modified creation operators to a vacuum vector. This space carries an irrep of the Lie algebra of SO(6, 2), so the model can aptly be called an SO(6, 2) model, just as Schwinger's model can be called a UW_2 model (see § 1). The question arises whether or not a *simple* model for SU(n), $n \neq 3$, can be obtained; in other words, can one find for each n an irrep of (the Lie algebra of) a corresponding non-compact, *simple* Lie group $K > \mathrm{SU}(n)$, which contains in direct sum each different irrep of SU(n) exactly once. Indeed, the question extends from SU(n) to other compact Lie groups. Positive answers can readily be given in some cases: for example any infinite-dimensional irrep of the homogeneous Lorentz group SO(3, 1) labelled $[0, c]$ in the usual $[k_0, c]$ notation, with c any non-integral complex number, contains every irrep of SO(3) exactly once (Gel'fand *et al* 1963) and so defines a model for that group.

Note that the irrep of the non-compact group defining the model need not be unitary, as this example shows: the irrep of SO(3, 1) is only unitary if c is pure imaginary, or lies in the interval [0, 1).

Another question concerns the realisation of a model, if it exists, in terms of boson operators. When this is possible, as in the SO(6, 2) model for SU(3), such a realisation has obvious computational advantages for the construction of the irreps of the compact group.

It has already been indicated above that the irrep of the SO(6, 2) algebra in $\mathscr{B}$ has an analogue for SO(4, 2). In fact it has such an analogue for SO($2n$, 2) $n = 1, 2, 4, 5, \ldots$. Simply take the range of free indices in (53) to be $1, 2, \ldots, n$ rather than 1, 2, 3; replace the subscripts 7 and 8 in (53) by $(2n+1)$, $(2n+2)$; replace the factor $(P+Q+1)$ in (39) and (40) by $(P+Q+n-2)$; and replace $(P+Q+2)$ in (51) by $(P+Q+n-1)$. Then a representation of the Lie algebra of SO($2n$, 2) on the Fock space $\mathscr{F}_{2n}$ is obtained, and it can be restricted to an irreducible subspace on which $\alpha_r\beta^r$ vanishes. Does this subrepresentation define a model for SU(n) when $n \neq 3$? The answer is surely no. Although the modified creation and annihilation operators are in each case Wigner operators for SU(n), it is not hard to see that for $n = 2$, irreps of SU(2) occur more than once, while for $n > 3$, some irreps of SU(n) do not occur at all. Thus SU(3) occupies a special place in this context, a rather surprising result.

In the case $n = 2$, there is a simple way to overcome this difficulty: restrict attention to the subspace obtained by applying to the vacuum vector, arbitrary polynomials in the $\bar{A}$-operators only (or the $\bar{B}$-operators only). This subspace carries every irrep of SU(2) exactly once, but it does not carry a representation of the full SO(4, 2) algebra. Instead, it carries an irrep of the subalgebra spanned by A_r, $\bar{A}^r$, and $A^r_s + \delta^r_s M$ ($M = P + Q + 1$ in this case). This is the Lie algebra of SU(2, 1), so it can be seen that there does exist a simple model for SU(2), namely an SU(2, 1) model, as well as Schwinger's UW_2 model which is not semi-simple. It could not be claimed that this SU(2, 1) model is as attractive as Schwinger's model from the point of view of constructing and analysing the representations of SU(2). It follows that simplicity of the non-compact group in terms of which a model for a compact subgroup is defined may not be an advantage.

In Schwinger's model, all Wigner tensor operators for SU(2) can be constructed very simply from the boson creation and annihilation operators. The situation is quite similar for the SO(6, 2) model for SU(3). Introduce the completely antisymmetric numerical SU(3) tensors ε^{rst}, ε_{rst}, with $\varepsilon^{123} = \varepsilon_{123} = +1$, and define

$$\varepsilon^r = \mathrm{i}\varepsilon^{rst}A_s\bar{B}_t, \qquad \varepsilon_r = \mathrm{i}\varepsilon_{rst}B^s\bar{A}^t, \tag{85}$$

which can be seen from (43) and (44) to be Hermitian conjugate to each other on $\mathscr{B}$. Then ε^r, like $\bar{A}^r$ and B^r, is a contravariant SU(3) vector (though, unlike $\bar{A}^r$ and B^r, it is *not* a contravariant U(3) vector), while ε_r, A_r and $\bar{B}_r$ are covariant SU(3) vectors. Thus, with T^r_s as in (9), one has

$$[\varepsilon^r, T^s_t] = -\delta^r_t\varepsilon^r + \tfrac{1}{3}\delta^s_t\varepsilon^r, \tag{86}$$

just as for $\bar{A}^r$ and B^r, and similarly

$$[\varepsilon_r, T^s_t] = \delta^s_r\varepsilon_t - \tfrac{1}{3}\delta^s_t\varepsilon_r \tag{87}$$

just as for A_r and $\bar{B}_r$.

The operators ε^r, $\bar{A}^r$ and B^r form a complete set (Louck and Biedenharn 1973) of contravariant SU(3)-vector Wigner operators, with the shifting values $(-1, +1)$, $(+1, 0)$

and $(0, -1)$ for the labelling operators (P, Q). (Recall that the SU(3) irreps are labelled $(p+q, q)$, where p and q are the eigenvalues of P and Q. Thus ε^r, for example, shifts the irrep $(p+q, q)$ to $(p+q, q+1)$.) Similarly ε_r, A_r and $\bar{B}_r$ form a complete set of covariant SU(3)-vector Wigner operators, with shifting values $(+1, -1)$, $(-1, 0)$ and $(0, +1)$ for (P, Q). Other simple vector operators which can be constructed, such as $A^r_s B^s$, turn out to be multiples (by SU(3) scalars) of these basic ones, in accordance with known general results (Green 1971). Note that since no SU(3) scalar can, on $\mathscr{B}$, fail to commute with P and Q, products like $\varepsilon^r A_r$, $B^r \varepsilon_r$ etc must vanish there.

Higher-rank irreducible tensor operators which are also Wigner operators can easily be constructed from the six basic vector operators. For example, nine obvious irreducible second-rank mixed tensor operators ('octet' operators) can be constructed. Together with their shifting values for (P, Q), they are

$$
\begin{aligned}
&\varepsilon^r A_s\ (-2, +1), \qquad && \varepsilon^r \bar{B}_s\ (-1, +2), \\
&\bar{A}^r \varepsilon_s\ (+2, -1), && \bar{A}^r \bar{B}_s\ (+1, +1), \\
&B^r \varepsilon_s\ (+1, -2), && B^r A_s\ (-1, -1), \\
&\varepsilon^r \varepsilon_s - \tfrac{1}{3}\delta^r_s(\varepsilon^t \varepsilon_t)\ (0, 0), && \\
&\bar{A}^r A_s - \tfrac{1}{3}\delta^r_s(\bar{A}^t A_t)\ (0, 0), && B^r \bar{B}_s - \tfrac{1}{3}\delta^r_s(B^t \bar{B}_t)\ (0, 0).
\end{aligned}
\tag{88}
$$

Of these, only eight are independent, as only two of the last three are independent. One finds that

$$
\begin{aligned}
\varepsilon^r \varepsilon_s - \tfrac{1}{3}\delta^r_s(\varepsilon^t \varepsilon_t) &= (P+Q+3)^2[G^r_s - \tfrac{1}{3}(P-Q+6)T^r_s], \\
\bar{A}^r A_s - \tfrac{1}{3}\delta^r_s(\bar{A}^t A_t) &= G^r_s + \tfrac{1}{3}(2P+Q-3)T^r_s, \\
B^r \bar{B}_s - \tfrac{1}{3}\delta^r_s(B^t \bar{B}_t) &= G^r_s - \tfrac{1}{3}(P+2Q+9)T^r_s,
\end{aligned}
\tag{89}
$$

where

$$
G^r_s = T^r_t T^t_s - \tfrac{1}{3}\delta^r_s(T^u_v T^v_u).
\tag{90}
$$

In fact any irreducible second-rank mixed tensor operator on $\mathscr{B}$ which has the shifting values $(0, 0)$ (i.e. one which commutes with P and Q and so leaves invariant each irrep of SU(3)) must be a linear combination (with SU(3)-scalar coefficients) of T^r_s and G^r_s, in accordance with a result obtained by Okubo (1962).

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